1 The Category of Graphs (or Pseudographs)

Definition 0.1

A *simple graph* $\mathcal{G}$ is an ordered pair of disjoint sets (N, E) of nodes $x \in N$ and edges $e_{xy} \in E$ such that E is a subset of the set $N^{(2)} = N \times N$ of unordered pairs of N. If the set N is finite then the graph $\mathcal{G}$ is also finite, as it is usually assumed with N and E being assumed to be finite, unless otherwise stated. The set N is the set of nodes, or *vertices*, and E is the set of *edges*.

Diagrams in a category can be considered as directed simple graphs in which the edges are replaced by arrows or morphisms that may satisfy commutativity:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[h]{} & D \end{array}$$

and identity conditions (or ETAC axioms).

Definition 0.2

A pseudograph $\mathcal{G}_P$ is an ordered triple (V, E, i), where V is a set called the *vertex set* of $\mathcal{G}_P$, E is a set called the *edge set* of G, and $i : E \to 2^V$ is the *incidence map*, such that for every $e_i \in E$, $1 \leq |i(e)| \leq 2$.

Remark: A pseudograph can be regarded as a generalization of the concept of graph.

Definition 0.3

For any two given pseudographs $G_{P1} = (V_1, E_1, i_1)$ and $G_{P2} = (V_2, E_2, i_2)$, a *graph homomorphism* h from G_{P1} to G_{P2} consists of two functions $f : V_1 \to V_2$ and $g : E_1 \to E_2$, such that

$$i_2 \circ g = f^* \circ i_1, \tag{1.1}$$

where the function $f^* : 2^{V_1} \to 2^{V_2}$ is defined as $f^*(S) = \{f(s) \mid s \in S\}$.

When $\mathcal{G}_1$ and $\mathcal{G}_2$ are just simple graphs, a graph homomorphism may be defined in terms of a single function $f : V_1 \to V_2$ satisfying the condition (*)

$$\{v_1, v_2\} \text{ is an edge of } \mathcal{G}_1 \Longrightarrow \{f(v_1), f(v_2)\} \text{ is an edge of } \mathcal{G}_2.$$

A *graph isomorphism* $h = (f, g)$ is a graph homomorphism such that both f and g are bijections.